The Internal Conflict of a Belief Function*

Johan Schubert

Abstract. In this paper we define and derive an internal conflict of a belief function We decompose the belief function in question into a set of generalized simple support functions (GSSFs). Removing the single GSSF supporting the empty set we obtain the base of the belief function as the remaining GSSFs. Combining all GSSFs of the base set, we obtain a base belief function by definition. We define the conflict in Dempster's rule of the combination of the base set as the internal conflict of the belief function. Previously the conflict of Dempster's rule has been used as a distance measure only between consonant belief functions on a conceptual level modeling the disagreement between two sources. Using the internal conflict of a belief function we are able to extend this also to non-consonant belief functions.

1 Introduction

In this paper we define and derive an internal conflict of a belief function within Dempster-Shafer theory [1–3, 14]. We decompose the belief function in question into a set of generalized simple support functions (GSSFs). Removing the single GSSF supporting the empty set we obtain the base of the belief function as the remaining GSSFs. Combining all GSSFs of the base set, we obtain a base belief function by definition. We define the conflict in Dempster's rule of this combination as the internal conflict of the belief function. We propose that the base belief function is a better measure than the original belief function which can be obtained by combining the base belief function with pure conflict, i.e., $\{[m_1(\emptyset), \emptyset],\ [m_1(\Theta), \Theta]\}$.

There are several different ways to manage a high conflict in combination of belief functions within Dempster-Shafer theory. For an overview of different

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alternatives to manage the combination of conflicting belief functions, see articles by Smets [16] and Liu [7]. For a recent survey of alternative distance between belief functions, see Jousselme and Maupin [5].

In section 2 we review a method for decomposing a belief function into a set of GSSFs [15]. In section 3 we derive the base set of a belief function and construct a base belief function from the base set corresponding to the belief function under decomposition. In section 4 we derive the internal conflict of the belief function and show how this extends the conflict from being a distance measure only for consonant belief functions to a functioning distance measure also between non-consonant belief functions. In section 5 we provide an example. Finally, conclusions are drawn (section 6).

2 Decomposing a Belief Function

All belief functions can be decomposed into a set of GSSFs on a frame of discernment Θ using the method developed by Smets [15]. A GSSF is either a traditional simple support function (SSF) [14] or an inverse simple support function (ISSF) [15]. Let us begin by defining an ISSF:

Definition 1. An inverse simple support function on a frame of discernment Θ is a function $m: 2^{\Theta} \rightarrow (-\infty, \infty)$ characterized by a weight $w \in (1, \infty)$ and a focal element $A \subseteq \Theta$, such that $m(\Theta) = w$, $m(A) = 1 - w$ and $m(X) = 0$ when $X \notin \{A, \Theta\}$.

Let us recall the meaning of SSFs and ISSFs [15]: An SSF $m_1(A) \in [0, 1]$ represents a state of belief that "You have some reason to believe that the actual world is in A (and nothing more)". An ISSF $m_2(A) \in (-\infty, 0)$ on the other hand, represents a state of belief that "You have some reason *not* to believe that the actual world is in A". Note that *not* believing in A is different than believing in A^c.

A simple example is one SSF $m_1(\mathrm{A}) = 1/4$ and $m_1(\Theta) = 3/4$, and one ISSF $m_2(\mathrm{A}) = -1/3$ and $m_2(\Theta) = 4/3$. Combining these two functions yields a vacuous belief function $m_{1 \oplus 2}(\Theta) = 1$.

The decomposition method is performed in two steps eqs. (1) and (2). First, for any non-dogmatic belief function Bel_0, i.e., where $m_0(\Theta) > 0$, calculate the commonality number for all focal elements A by eq. (1). We have

$$Q_0(A) = \sum_{B \supseteq A} m_0(B) \tag{1}$$

For dogmatic belief functions assign $m_0(\Theta) = \varepsilon > 0$ and discount all other focal elements proportionally.

Secondly, calculate $m_i(C)$ for all decomposed GSSFs, where $C \subseteq \Theta$ including $C = \varnothing$, and i is the ith GSSF. There will be one GSSF for each subset C of the frame unless $m_i(C)$ happens to be zero. In the general case we will have $|2^\Theta|$ GSSFs. We get for all $C \subseteq \Theta$ including $C = \varnothing$

$$\begin{aligned} m_i(C) &= 1 - \prod_{A \supseteq C} Q_0(A)^{(-1)^{|A|-|C|+1}} \\ m_i(\Theta) &= 1 - m_i(C) \end{aligned} \tag{2}$$

where $i \in [1, 2^{|\Theta|} - 1]$.

Here, $C \subseteq \Theta$ of $m_i(C)$ is the *i*th subset of Θ in numerical order[2] which also includes $C = \varnothing$, i.e., $\{[m_1(\varnothing), \varnothing], [m_1(\Theta), \Theta]\}$ is the first decomposed GSSF of eq. (2).

3 Transforming a Belief Function into a Base Belief Function

Using eqs. (1) and (2) we may decompose a belief function m_0 into a set of GSSFs. We call the non-conflict GSSFs of the decomposition the base of m_0.

Definition 2. The base of a belief function m_0 is the set of decomposed simple support and inverse simple support function

$$\{m_i\}_{i=2}^{|2^\Theta|-1} \tag{3}$$

deliberately excluding m_1 that supports only $\{\varnothing, \Theta\}$, where

$$\{m_i\}_{i=1}^{|2^\Theta|-1} \tag{4}$$

is the full set of $|2^\Theta| - 1$ simple support and inverse simple support function from the decomposition of m_0 by eqs. (1) and (2).

Definition 3. A base belief function m_{00} of a belief function m_0 is the belief function resulting from the unnormalized combination of the base of m_0, i.e.,

$$m_{00} = \bigcirc \{m_i\}_{i=2}^{|2^\Theta|-1} . \tag{5}$$

Definition 4. A base conflict of a belief function m_0 is the obtained conflict $m_1(\varnothing)$ of the first GSSF that supports $\{[m_1(\varnothing), \varnothing], [m_1(\Theta), \Theta]\}$ of the decomposition of a belief function of m_0 by eqs. (1) and (2).

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[2] With $\Theta = \{a, b, c\}$ the numerical order of all subsets in Θ including $\varnothing$ is $\Theta = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

Theorem 1. *A belief function m_0 can be recovered by combination of its corresponding base belief function m_{00} with the base conflict m_1, i.e.,*

$$m_0 = m_{00} \bigcirc m_1 \tag{6}$$

Proof. Immediate by definition 2 and 3. □

4 The Internal Conflict of a Belief Function

The conflict received from a combination of belief functions by Dempster's rule is not a measure of dissimilarity between the combined belief functions. Indeed, belief functions can be quite different and yet have zero conflict as intersection of their focal elements are non-empty. Instead the conflict of Dempster's rule is best viewed as a different kind of distance measure; a measure of conceptual disagreement between sources. When they disagree highly it is a sign that something is wrong. It should be noted that there is at least two possible sources of conflict other than measurement errors. We may have modeling errors or faulty sources [4]. Faulty sources are corrected by appropriate discounting (e.g., [6, 12, 16]) while modeling errors are corrected by adopting an appropriate frame of discernment [13, 14].

In this section we define and investigate an internal conflict of a belief function. We further devise a way to obtain the internal conflict.

Definition 5. The internal conflict of a belief function m_0 is the conflict received in the unnormalized combination of the base of m_0 to obtain the base belief function m_{00}, i.e., $m_{00}(\varnothing)$ where

$$m_{00}(\varnothing) = \bigcirc\{m_i\}_{i=2}^{|2^\Theta|-1}(\varnothing). \tag{7}$$

For simplicity, view the intersection of eq. (7) as taking place in a $|2^\Theta|-2$ hyper cube of all $|2^\Theta|-2$ GSSFs. Note that $m_{00}(\varnothing)$ can take both positive and negative values.

Theorem 2. The internal conflict within a base belief function is strictly a function of conflicts between different GSSFs supporting subsets of the frame.

Proof. Immediate by observation of the combination in eq. (7) as m_1 with body of evidence $\{[m_1(\varnothing), \varnothing], [m_1(\Theta), \Theta]\}$ is not included in the combination. □

Theorem 3. There exist an infinite size family of unnormalized belief functions $\{ m_0^p | p = m_1(\varnothing) \in (-\infty, \infty), p \neq 1 \}$ with an identical base belief function m_{00} and identical internal conflict.

Proof. Let us generate a family from the base: Take any belief function m_0 on a frame Θ of size $n = |\Theta|$. Decompose m_0 into its 2^n GSSFs. Combine $\{m_i\}_{i=2}^{2^n}$ using eq. (7) into the base belief function. Let us ignore the value obtained for $m_1(\varnothing)$ in the decomposition. Instead, let $m_1(\varnothing) \in (-\infty, \infty)$, $m_1(\varnothing) \neq 1$. The family of belief functions $\{m_0^p\}$ is generated by combining the base belief function m_{00} with each of the $\{m_1\}$. The family is of infinite size. □

When going in the other direction from family to base: Note, that in the special case of a normalized base belief function it can be recovered from any family member by normalization.

If we combine a non-consonant belief function with itself we should not be surprised that we receive a conflict. A non-consonant belief function can be expressed as a construct from the base set of that belief function. If the belief function combined with itself is constructed in two steps by first combining all GSSFs pairwise with themselves, these $|2^\Theta| - 2$ combinations of GSSFs with identical focal sets are conflict free (excluding $\varnothing$ and Θ). Combining the resulting $|2^\Theta| - 2$ GSSFs in the second step obviously has empty intersections among their focal elements resulting in conflict. Thus, the internal conflict received is a function of conflicts from different GSSFs (excluding $m_1(\varnothing)$) that are used to construct the non-consonant belief function. Thus, the conflict noticed in the combination exists internally within non-consonant belief functions before combination and is a consequence of the scattering of mass within the distribution. This makes the internal conflict appropriate as a conceptual distance measure also between non-consonant belief functions as it measures the internal conflict in the combination of GSSFs from two different base sets corresponding to the two different base belief functions without the added pure conflict of m_1 (supporting only $\varnothing$ and Θ) that is always included in the conflict obtained by Dempster's rule.

Thus, from theorem 2 and 3 follows that the internal conflict is an appropriate distance measure for all belief functions as it excludes the pure conflict of $m_1(\varnothing)$ (i.e., also for non-consonant belief functions), where this distance measure sought after is a measure of conceptual disagreement between sources.

When calculating the conceptual distance measure based on internal conflict between two belief functions we first transform the two belief functions m_0 and m_0' to their base belief function form using eqs. (1) and (2) to find the base set, this is followed by eq. (5) to construct the base belief function, m_{00} and m_{00}'. We perform a conjunctive combination $m_{00}'' = m_{00} \bigcirc\!\!\!\!\cap\, m_{00}'$ and find the internal conflict $m_{00}''(\varnothing)$ of the resulting base belief function using eq. (7).

This measure $m''_{00}(\varnothing)$ of internal conflict is the most objective conflict measure since it excludes pure conflict and is immune to normalizations and incoming belief functions from sources without any information on conflicts in earlier combinations.

In the problem of partitioning mixed-up belief functions into subsets that correspond to different subproblems [8–11] we may use the distance measure of internal conflict for all belief functions (i.e., also for non-consonant belief functions).

5 An Example

Let us study a simple example. In this example we will represent all belief functions using numerical ordering of focal elements.

We assume a frame of discernment $\Theta = \{a, b\}$ and a belief function m_0 that is build up by combination of two SSFs m_2 and m_3 that are yet unknown to us, where

$$m_2 = [0, 0.4, 0, 0.6], \qquad m_3 = [0, 0, 0.4, 0.6]. \tag{8}$$

We have,

$$m_0 = m_2 \bigcirc m_3 = [0.16, 0.24, 0.24, 0.36]. \tag{9}$$

Using eqs. (1) and (2), m_0 can be decomposed into the base SSFs m_2 and m_3. Here, the base conflict is 0, i.e., m_1 in the decomposition of m_0 is a vacuous SSF; $m_1(\Theta) = 1$. From the base set eq. (8) we can construct the base belief function m_{00} which in this case is identical to the belief function m_0.

If m_0 is normalized then the situation is different. Let us call this normalization m_{0n}. We have,

$$m_{0_n} = [0, 0.2857, 0.2857, 0.4286]. \tag{10}$$

Decomposing m_{0n} we get

$$\begin{aligned} &m_{1_n} = [-0.1905, 0, 0, 1.1905], \qquad m_{2n} = [0, 0.4, 0, 0.6], \\ &m_{3_n} = [0, 0, 0.4, 0.6] \end{aligned} \tag{11}$$

which is the same base for m_{0n} as in the decomposition of m_0. Thus, m_0 and m_{0n} has the same base belief function, which is $m_{00} = m_0$. However, we obtain an inverse base conflict of $m_{1_n}(\varnothing) = -0.1905$ when decomposing m_{0n} compared to $m_1(\Theta) = 1$ in the decomposition of m_0.

Furthermore, let us assume that we have a second belief function m'_0 which is build up by combination of two SSFs m_2 and m_3 that are also unknown to us, where

$$m'_2 = [0, 0.3, 0, 0.7], \qquad m'_3 = [0, 0, 0.3, 0.7]. \tag{12}$$

We have

$$m_0' = m_2' \mathbin{\bigcirc\mkern-15mu\cap} m_3' = [0.09, 0.21, 0.21, 0.49]\,. \tag{13}$$

As above, using eqs. (1) and (2) m_0' can be decomposed into the base m_2' and m_3' (m_1' is vacuous). Using eq. (5) we construct the base belief function m_{00}'.

Finally, given both m_{00} and m_{00}' we combine them to obtain

$$m_{00}'' = m_{00} \mathbin{\bigcirc\mkern-15mu\cap} m_{00}' = [0.3364, 0.2436, 0.2436, 0.1764]\,. \tag{14}$$

We notice a conflict of 0.3364 in the combination of m_{00} and m_{00}'.

Assuming instead that we receive m_{00}'' from a source (let us then call it m_0'') we can decompose it to obtain a pure base without any base conflict;

$$\begin{aligned} &m_1'' = [0, 0, 0, 1], \qquad m_3'' = [0, 0.58, 0, 0.42], \\ &m_3'' = [0, 0, 0.58, 0.42]. \end{aligned} \tag{15}$$

We should notice that the two base SSFs m_2'' and m_3'' are themselves conflict free combinations $m_2 \mathbin{\bigcirc\mkern-15mu\cap} m_2'$ and $m_3 \mathbin{\bigcirc\mkern-15mu\cap} m_3'$ resulting in m_2'' and m_3'', respectively.

Recombining m_2'' and m_3'' yields a base belief function m_{00}'' identical to m_0''. Thus, the conflict of m_0'' and the internal conflict of m_{00}'' are identical in this case.

Had m_0'' been normalized the situation is somewhat different. Let us call the normalization m_{0n}'''. We have,

$$m_{0n}''' = [0, 0.3671, 0.3671, 0.2658]. \tag{16}.$$

It can be decomposed into

$$\begin{aligned} &m_{1n}'' = [-0.3082, 0, 0, 1.3082], \qquad m_{2n}'' = [0, 0.58, 0, 0.42], \\ &m_{3n}'' = [0, 0, 0.58, 0.42]. \end{aligned} \tag{17}$$

We observe that m_0'' and m_{0n}'' have the same base set in that $m_2'' = m_{2n}''$ and $m_3'' = m_{3n}''$.

Finally, let us study a combination of a belief function with itself. We combine m_{0n}''' with itself. We have,

$$m_{0n}'''' = m_{0n}''' \mathbin{\bigcirc\mkern-15mu\cap} m_{0n}''' = [0.2695, 0.3299, 0.3299, 0.0706] \tag{18}$$

where $m''''_{0n}(\varnothing) = 0.2695$ is the internal conflict distance measure between the two belief function.

Decomposing m''''_{0n} we get,

$$
\begin{aligned}
& m''''_{1n} = [-1.2711, 0, 0, 2.2711] \qquad m''''_{2n} = [0, 0.8236, 0, 0.1764], \\
& m''''_{3n} = [0, 0, 0.8236, 0.1764].
\end{aligned}
\tag{19}
$$

As before we have a base set of two SSFs. Here the base set of m''''_{0n} is m''''_{2n} and m''''_{3n}, where

$$
m''''_{2n} = m''_{2n} \textcircled{\scriptsize$\cap$} m''_{2n}, \qquad m''''_{3n} = m''_{3n} \textcircled{\scriptsize$\cap$} m''_{3n}. \tag{20}
$$

6 Conclusions

We conclude that the internal conflict of a non-consonant belief function is actually a function of conflicts between different GSSFs in the base set of that belief function. Here all GSSFs that have identical focal elements have zero conflict. Thus, the internal conflict between two belief functions is an appropriate distance measure on a conceptual level that measures disagreement between sources.

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